SPARSE METRIC SPACES AND SPARSE ENDS

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Abstract. We study metric spaces that in some sense thin out at infinity. We define and investigate a measure of sparsity that is a quasi-isometry invariant, and introduce an analogue of topological ends for sparse spaces that is also invariant under quasi-isometries. We study some examples arising in various contexts.

1. Introduction

We study metric spaces that in some sense thin out at infinity. While our motivation comes from the search for invariants in the theory of coarse dynamical systems ([GM1]), such spaces arise in diverse areas, including the study of Roe algebras ([BCL], [BFV], [CMS], [WW]), substitutions and other symbolic dynamical systems ([LM]), and number theory ([M], [Z]).

Our primary goals here are to define and analyze some closely related notions of thinning out at infinity, give a measure of sparsity that is a quasi-isometry invariant, and define an analogue of topological ends for sparse spaces that is also invariant under quasi-isometries.

Protasov [P] has previously studied sparse metric spaces from a different point of view. Although his definition of sparse spaces differs from ours, his unbounded thin spaces are easily seen to be equivalent to our sparse spaces, and his coarsely thin spaces coincide with our quasi-sparse spaces. In particular, our Proposition 2.6 follows from his Theorem 5, and our Proposition 2.17 from his Theorem 1 and the remarks before it.

In Section 2, we define sparse metric spaces and the related notions of quasi-sparse and finitely sparse spaces, introduce the order of sparsity and observe that it is invariant under quasi-isometries, study the relation of quasi-sparsity to asymptotic dimension, and present a number of examples and connections to recent work in other fields.

Section 3 draws a connection between sparsity of a metric space and the notion of tryposity at infinity, which involves the degree to which pairs of points in large balls fail to have midpoints. The subsequent sections do not depend on it.

In Section 4 we define for each $\alpha \geq 0$ the space of α-ends of a metric space, and show that a quasi-isometry of metric spaces induces a homeomorphism on their spaces of α-ends. When $\alpha = 0$ this recovers the coarse ends studied in [AC], while for $\alpha > 0$ this gives a useful notion of ends for sparse spaces.

In Section 5 we give a number of examples of metric spaces and their spaces of α-ends.

Date: June 9, 2026.

2010 *Mathematics Subject Classification.* Primary 51F30; secondary 53C23, 11A41, 11B05, 46L05, 37B10.

We thank Slawek Klimek and Jon Keating for useful suggestions and references.

2. Sparse metric spaces

Let (X, d) be an unbounded metric space. Fix a base point $x_0 \in X$. Define a function $s : (0, \infty) \to \mathbb{R}$ as follows:

$$s(R) = \inf\{d(x, y) : x, y \in X \setminus B(x_0, R), x \neq y\},$$

where $B(x_0, R)$ is the open ball of radius R, centered at x_0. Clearly, the function s is nondecreasing. We will assume that $\lim_{r\to\infty} s(R) = \infty$ and will call such spaces *sparse at infinity*, or just *sparse*. This property does not depend on the choice of x_0, since for $x_1, x_2 \in X$ we have $B(x_2, R) \subset B(x_1, R + d(x_1, x_2))$.

Now we define the *lower and upper sparsity order at infinity* ($\underline{\sigma}$ and $\overline{\sigma}$) as

$$\underline{\sigma}(X) = \varliminf_{R\to\infty} \frac{\log s(R)}{\log R}, \qquad \overline{\sigma}(X) = \varlimsup_{R\to\infty} \frac{\log s(R)}{\log R}.$$

If $\underline{\sigma}(X) = \overline{\sigma}(X)$, we will speak simply of the *sparsity order at infinity* or just the *sparsity order* and denote it $\sigma(X)$.

These similarly do not depend on the choice of x_0.

Let us give some examples. In these examples $X = \{0\} \cup \{a_n : n = 1, 2, 3, \dots\} \subset \mathbb{R}$.

Example 2.1. For each $q > 1$, define X_q by taking $a_n = n^q$. In order to compute $\underline{\sigma}(X_q)$ we take $R = n^q$:

$$\underline{\sigma}(X_q) = \lim_{n\to\infty} \frac{\log((n + 1)^q - n^q)}{\log n^q}.$$

We have

$$\lim_{x\to\infty} \frac{(x + 1)^q - x^q}{x^{q-1}} = \lim_{x\to\infty} \frac{(1 + 1/x)^q - 1}{1/x} = \lim_{y\to 0^+} \frac{(1 + y)^q - 1}{y} = q.$$

Therefore

$$\underline{\sigma}(X_q) = \lim_{n\to\infty} \frac{\log(qn^{q-1})}{\log n^q} = \frac{q - 1}{q}.$$

In order to compute $\overline{\sigma}(X_q)$ we take $R = n^q + \varepsilon$ and then the limit as $\varepsilon \to 0$:

$$\overline{\sigma}(X_q) = \lim_{n\to\infty} \frac{\log((n + 2)^q - (n + 1)^q)}{\log n^q}.$$

Similarly as before, we get

$$\overline{\sigma}(X_q) = \lim_{n\to\infty} \frac{\log(q(n + 1)^{q-1})}{\log n^q} = \frac{q - 1}{q}.$$

Thus, $\sigma(X_q) = 1 - 1/q$. ♢

Example 2.2. Now we consider the case of $a_n = 2^n$. We get

$$\underline{\sigma}(X) = \lim_{n\to\infty} \frac{\log(2^{n+1} - 2^n)}{\log 2^n} = 1$$

and similarly,

$$\overline{\sigma}(X) = \lim_{n\to\infty} \frac{\log(2^{n+2} - 2^{n+1})}{\log 2^n} = 1.$$

Thus, $\sigma(X) = 1$. ♢

Example 2.3. This example is a sparse space with sparsity order $\sigma = 0$.

Let $a_n = n \log n$. To compute $\overline{\sigma}(X)$, we take $R = n \log n + \varepsilon$ and then as before the limit as $\varepsilon \to 0$. Note that

$$\lim_{x\to\infty} \frac{(x+1)\log(x+1) - x\log x}{\log x} = \lim_{x\to\infty} \frac{\log(1+1/x)}{1/x} = \lim_{y\to 0^+} \frac{\log(1+y)}{y} = 1.$$

Therefore

$$\overline{\sigma}(X) = \lim_{n\to\infty} \frac{\log[(n+2)\log(n+2) - (n+1)\log(n+1)]}{\log(n\log n)} = \lim_{n\to\infty} \frac{\log\log(n+1)}{\log(n\log n)} = 0.$$

Thus, $\sigma(X) = \overline{\sigma}(X) = 0$. $\Diamond$

Example 2.4. Take $a_n = 2^{b^n}$ for some $b > 1$. We have

$$\underline{\sigma}(X) = \lim_{n\to\infty} \frac{\log(2^{b^{n+1}} - 2^{b^n})}{\log 2^{b^n}} = \lim_{n\to\infty} \frac{b^{n+1}}{b^n} = b.$$

Similarly,

$$\overline{\sigma}(X) = \lim_{n\to\infty} \frac{\log(2^{b^{n+2}} - 2^{b^{n+1}})}{\log 2^{b^n}} = \lim_{n\to\infty} \frac{b^{n+2}}{b^n} = b^2.$$

$\Diamond$

It is easy to see that similar computations for the triple exponential for a_n give $\sigma(X) = \infty$.

Sparsity order at infinity is invariant under quasi-isometry. A map $\varphi : X \to Y$ between metric spaces is called a (C, A)-*quasi-isometric embedding* if for every $x, x' \in X$

$$C^{-1}d(x,x') - A \le d(\varphi(x), \varphi(x')) \le Cd(x,x') + A.$$

If in addition Y is contained in an A-neighborhood of $\varphi(X)$, i.e. for every $y \in Y$ there is $x \in X$ with $d(y, \varphi(x)) \le A$, φ is called a (C, A)-*quasi-isometry,* in which case X and Y are said to be *quasi-isometric.*

More generally, a map $\varphi : X \to Y$ between metric spaces is called a *coarse embedding* if the affine maps in the inequality are replaced by arbitrary unbounded nondecreasing maps $\rho_-, \rho_+ : [0,\infty) \to [0,\infty)$ respectively. Such a φ is a *coarse equivalence* if in addition Y is contained in some A-neighborhood of $\varphi(X)$.

Note that the quasi-isometric image of a sparse metric space is sparse, and more generally the image by a coarse embedding of a sparse metric space is sparse.

Proposition 2.5. *If metric spaces X and Y are sparse at infinity and quasi-isometric and $\sigma(X)$ exists, then $\sigma(Y)$ exists and $\sigma(Y) = \sigma(X)$.*

Proof. Let $\varphi : X \to Y$ be a (C, A)-quasi-isometry. Then for $d(x, x')$ sufficiently large, φ is $D := C + 1$ bi-Lipschitz:

$$D^{-1}d(x,x') \le d(\varphi(x), \varphi(x')) \le Dd(x,x').$$

Then for sufficiently large R,

$$s_Y(R/D) \le Ds_X(R)$$

so that

$$\overline{\lim_{R\to\infty}}\, s_Y(R/D) \le \overline{\lim_{R\to\infty}}\, Ds_X(R)$$

and so

$$\overline{\sigma}(Y) \leq \sigma(X).$$

Similarly, for D' a large scale bi-Lipschitz constant for an inverse quasi-isometry φ',

$$s_X(R/D') \leq D' s_Y(R)$$

so that

$$\underline{\lim}_{R\to\infty} s_X(R/D') \leq \underline{\lim}_{R\to\infty} D' s_Y(R)$$

and so

$$\sigma(X) \leq \underline{\sigma}(Y).$$

So

$$\underline{\sigma}(Y) = \overline{\sigma}(Y) = \sigma(Y) = \sigma(X).$$

$\square$

In contrast to this result, note that coarse equivalence of sparse metric spaces does not preserve sparsity order. For example, the subspaces of the halfline $X_2 = \{0\} \cup \{n^2 | n \in \mathbb{N}\}$ and $X_3 = \{0\} \cup \{n^3 | n \in \mathbb{N}\}$ are coarsely equivalent, for example with $\rho_-(x) = x$ and $\rho_+(x) = x^2$, while $\sigma(X_2) = 1/2$ and $\sigma(X_3) = 2/3$.

We want to extend the sparsity order at infinity to spaces that are quasi-isometric to spaces that are sparse at infinity. If a metric space X has a sparse subspace X' such that X is contained in some A-neighborhood of X', we call the subspace X' a *sparse skeleton* of the space X. Clearly, any two sparse skeletons of a metric space are quasi-isometric.

Proposition 2.6. (cf. [P]). *The following are equivalent for a metric space X :*
(1) *X is coarsely equivalent to a sparse space.*
(2) *X is quasi-isometric to a sparse space.*
(3) *X has a sparse skeleton.*

Proof. Trivially (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). But also (1) $\Longrightarrow$ (3): Let φ be a coarse equivalence from a sparse space Y to X. Then $X' := \varphi(Y)$ is a sparse subspace of X and X is contained in some A-neighborhood of X'. $\square$

Definition 2.7. We call a metric space satisfying the conditions of Proposition 2.6 *quasi-sparse.*

Note that an unbounded subspace of a sparse metric space is sparse, and similarly an unbounded subspace of a quasi-sparse space is quasi-sparse.

Remark 2.8. If X is quasi-sparse, in light of Proposition 2.5 we can define its lower and upper sparsity order at infinity as that of any of its sparse skeletons, and similarly for its sparsity order at infinity.

The following definition appears in [BCL], where it is called sparse rather than finitely sparse and is used in connection with the rigidity of Roe algebras; see also [BFV] and [CMS].

Definition 2.9. Let (X, d) be a metric space. It is *finitely sparse* if there exists a partition $X = \bigsqcup_{n=1}^{\infty} X_n$, such that the cardinality of each X_n is finite and $d(X_n, X_m) \to \infty$ as $n + m \to \infty$.

Clearly a finitely sparse metric space for which the X_n can be taken to be singletons must be sparse, and more generally one for which they can be taken to be uniformly bounded in diameter must be quasi-sparse.

In the other direction, a quasi-sparse locally finite metric space is finitely sparse. (Recall that a locally finite metric space is one in which every ball has finite cardinality.)

Example 2.10. Consider the one-sided full shift on two symbols

$$\Sigma_2 = \{0,1\}^{\mathbb{N}}.$$

We can view a point $x \in \Sigma_2$ as the subspace of the metric space $\mathbb{N}$ determined by the locations of the 1's in x. If

$$\sigma : \Sigma_2 \to \Sigma_2$$

is the left shift

$$\sigma(x_1 x_2 x_3 \ldots) = (x_2 x_3 x_4 \ldots),$$

then x is sparse (respectively quasi-sparse, finitely sparse) if and only if every point in its σ-orbit is also sparse (respectively quasi-sparse, finitely sparse). $\Diamond$

Example 2.11. Let $S \subset \Sigma_2$ be the subshift consisting of those sequences in which the number of consecutive 0's preceding successive 1's is strictly increasing. Then every $s \in S$ with infinitely many 1's is sparse. $\Diamond$

Example 2.12. Consider the Cantor substitution $0 \longrightarrow 000,\ 1 \longrightarrow 101$, starting with 1. (Here the initial segment of length 3^n shows which triadic subintervals of the unit interval are present in the n-th stage of the construction of the Cantor set.) The set of integers corresponding to the resulting sequence

$$101000101000000000101000101\ldots$$

is finitely sparse but not quasi-sparse. $\Diamond$

Example 2.13. Consider the two-dimensional Sierpiński substitution taking 0 to a 3 by 3 block of 0's and 1 to a 3 by 3 block with 1's in the corners and 0's elsewhere, starting with 1. The set of points in the positive quadrant of the integer lattice in the plane corresponding to this substitution is similarly finitely sparse but not quasi-sparse. $\Diamond$

Example 2.14. Let P be the set of prime numbers. Then Zhang's celebrated result on bounded gaps between primes [Z] shows that P is not sparse. On the other hand, the fact that there are successive primes with arbitrarily long gaps between them implies that P is finitely sparse. But as the following proposition shows, it follows easily from results of Maynard on the heels of Zhang that P is not quasi-sparse. $\Diamond$

Proposition 2.15. *The primes are not quasi-sparse.*

Proof. If P were quasi-sparse, there would be a sparse skeleton $\{p_{s_i}\}$ and an $a > 0$ such that $p_{s_{i+1}} - p_{s_i} \longrightarrow \infty$, and for each $p \in P$ there would exist $s_i =: s(p)$ with $|p - p_{s_i}| \leq a$.

Fix $m > 2a$. Then by Maynard [M], $\liminf(p_{n+m} - p_n) < \infty$, so there exists a sequence $\{n_i\}$ such that $(p_{n_i+m} - p_{n_i}) \longrightarrow M < \infty$.

But $p_{n_i+m} - p_{n_i} \geq m > 2a$, so $s_i' := s(p_{n_i+m}) > s_i := s(p_{n_i})$.

So $p_{n_i+m} - p_{n_i} = p_{n_i+m} - p_{s_i'} + p_{s_i'} - p_{s_i} + p_{s_i} - p_{n_i} \geq p_{s_i'} - p_{s_i} - 2a \longrightarrow \infty$. $\square$

Gromov [G] introduced the notion of asymptotic dimension (see also [BD]) originally as a geometric invariant for finitely generated groups.

Definition 2.16. Let (X, d) be a metric space. X has *asymptotic dimension zero* if for every $r > 0$, there is a partition of X into uniformly bounded r-disjoint sets.

Here a family of sets is r-disjoint if every distinct pair of sets is at distance larger than r.

Proposition 2.17. (cf. [P]). *If a metric space X is quasi-sparse, then it has asymptotic dimension zero.*

Proof. If X is sparse, given r a partition consisting of all singletons except for a single sufficiently large ball works. Since a quasi-sparse space is coarsely equivalent to a sparse space and asymptotic dimension is a coarse invariant, quasi-sparse spaces also have asymptotic dimension zero. □

On the other hand, a metric space of asymptotic dimension zero need not be quasi-sparse, since in particular, the product of sparse spaces is not quasi-sparse, while it has asymptotic dimension zero as the product of spaces with asymptotic dimension zero [BD]:

Proposition 2.18. *If X and Y are sparse spaces, then $X \times Y$ is not quasi-sparse.*

Proof. Suppose $X \times Y$ is quasi-sparse. Then it has a skeleton Z. This Z is A-dense in $X \times Y$ for some $A > 0$. Choose $y_0 \in Y$. If $d(y_0, y)$ is sufficiently large, then $Z \cap (X \times \{y\})$ is A-dense in $X \times \{y\}$. There exist $x_1, x_2 \in X$ (independent of y) such that $d(x_1, x_2) > 2A$. Then there are two distinct elements of $Z \cap (X \times \{y\})$ whose distance from one another is at most $d(x_1, x_2) + 2A$. This happens for every y for which $d(y_0, y)$ is sufficiently large, so Z is not sparse, a contradiction. □

3. Sparsity and tryposity at infinity

In [GM2], the authors introduced the *convexity gap* in metric spaces, and used its local behavior to study curvature. In this section, we relate sparsity to the large scale behavior of this nonconvexity measure in metric spaces. The subsequent sections of this paper are independent of this section.

Let (X, d) be a metric space, $x_0 \in X$, and $\overline{B}(x, R)$ be the closed ball in X, centered at $x \in X$ and of radius R. For $x, y \in X$ we define the *hyperdistance* between x and y as

$$hd(x, y) = 2\ \inf\{r : \overline{B}(x, r) \cap \overline{B}(y, r) \neq \emptyset\}.$$

Next, we define the *leipodistance* between x and y by

$$ld(x, y) = hd(x, y) - d(x, y).$$

Observe that $d(x, y) \leq hd(x, y) \leq 2d(x, y)$, so $0 \leq ld(x, y) \leq d(x, y)$. The leipodistance between a pair of points is a measure of the degree of their failure to have a midpoint.

Now we define the *convexity gap* (or simply *gap*) of X as

$$\gamma(X) = \sup\{ld(x, y) : x, y \in X\}.$$

In [GM2], the authors studied how the gap of a small ball around a point on a curve shrank as its radius shrank, and related this *tryposity* to curvature. In contrast, here we will look at how the gap of a large ball grows as its radius grows, and relate the resulting notion of tryposity at infinity to sparsity.

Define the *lower and upper tryposity order at infinity* ($\underline{\tau}$ and $\overline{\tau}$) as

$$\underline{\tau}(X) = \varliminf_{R\to\infty} \frac{\log \gamma(\overline{B}(x_0, R))}{\log R}, \qquad \overline{\tau}(X) = \varlimsup_{R\to\infty} \frac{\log \gamma(\overline{B}(x_0, R))}{\log R}.$$

If $\underline{\tau}(X) = \overline{\tau}(X)$, we will speak simply of the *tryposity order at infinity* and denote it $\tau(X)$. These do not depend on the choice of x_0.

We revisit the first two examples of the previous section, where $X = \{0\} \cup \{a_n : n = 1, 2, 3, \dots\} \subset \mathbb{R}$.

Example 3.1. Fix $q > 1$. Then X_q takes $a_n = n^q$. In order to compute $\overline{\tau}(X_q)$ we take $R = (n+1)^q$:

$$\overline{\tau}(X) = \varlimsup_{R\to\infty} \frac{\log(\overline{B}(x_0, R))}{\log R} = \lim_{n\to\infty} \frac{\log((n+1)^q - n^q)}{\log(n+1)^q}.$$

We use that

$$\lim_{x\to\infty} \frac{(x+1)^q - x^q}{(x+1)^{q-1}} = \lim_{x\to\infty} \frac{(x+1)^q - x^q}{x^{q-1}} = q$$

as bef

Therefore

$$\overline{\tau}(X_q) = \lim_{n\to\infty} \frac{\log((n+1)^q - n^q)}{\log(n+1)^q} = \frac{q-1}{q}.$$

In order to compute $\underline{\tau}(X_q)$ we take $R = (n+1)^q - \varepsilon$ and then the limit as $\varepsilon \to 0$:

$$\underline{\tau}(X_q) = \lim_{n\to\infty} \frac{\log(n^q - (n-1)^q)}{\log(n+1)^q}.$$

Similarly as before, we get

$$\underline{\tau}(X_q) = \lim_{n\to\infty} \frac{\log(q(n-1)^{q-1})}{\log(n+1)^q} = \frac{q-1}{q}.$$

Thus, $\tau(X_q) = 1 - 1/q$. ◊

Example 3.2. Now we consider the case of $a_n = 2^n$. We get

$$\overline{\tau}(X) = \lim_{n\to\infty} \frac{\log(2^{n+1} - 2^n)}{\log 2^{n+1}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

and similarly,

$$\underline{\tau}(X) = \lim_{n\to\infty} \frac{\log(2^n - 2^{n-1})}{\log 2^{n+1}} = \lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

Thus, $\tau(X) = 1$. ◊

Remark 3.3. If a metric space X is contained in some $A-$neighborhood of a subspace X', then $\overline{\tau}(X') = \overline{\tau}(X)$ and $\underline{\tau}(X') = \underline{\tau}(X)$ so that $\tau(X') = \tau(X)$ if they exist.

In general, the order of tryposity at infinity is not invariant under quasi-isometry:

Example 3.4. (The Mountain Range) Let M be the subset of the plane consisting of the origin and segments from $(2^n, 0)$ to $((3/2)2^n, 2^n)$ and from $((3/2)2^n, 2^n)$ to $(2^{n+1}, 0)$ for $n \in \mathbb{Z}$. Then $\tau(M) = 1$. But M is quasi-isometric to the nonnegative horizontal axis since projection onto the horizontal coordinate is a bi-Lipschitz map from M onto the nonnegative reals, and the halfline has $\tau = 0$.

The following modification of Example 3.4 shows that even among sparse spaces the tryposity order at infinity is not a quasi-isometry invariant.

Example 3.5. Consider the subspace M' of M consisting of the origin and the endpoints of the segments of M and $2^{n/2}$ equally spaced points in the interior of the segment starting at 2^n and the segment ending at 2^{n+1} on the x-axis. Then M' is sparse, with $\sigma(M') = 1/2$, and its tryposity order at infinity is $\tau(M') = \tau(M) = 1$, but its image in the x-axis by the bi-Lipschitz projection map has $\tau = 1/2$.

Proposition 3.6. *If X is quasi-sparse of order $0 < \sigma \leq 1$ then the upper tryposity order at infinity $\overline{\tau}(X) \geq \sigma(X)$.*

Proof. By Remark 3.3, it suffices to prove the proposition for a sparse skeleton of a given quasi-sparse space. So we assume X is sparse of order σ.

For all $x' \in X \setminus B(x_0, R + s(R))$, and for all $x \in X \setminus B(x_0, R)$, we have $d(x, x') \geq s(R)$. So for all $x', x'' \in X \setminus B(x_0, R + s(R))$, we have $ld(x', x'') \geq s(R)$. Thus $\gamma(\overline{B}(x_0, R + s(R))) \geq s(R)$, and

$$\frac{\log \gamma(\overline{B}(x_0, R + s(R)))}{\log(R + s(R)} \geq \frac{\log s(R)}{\log(R + s(R)} = \frac{\log s(R)}{\log R} \frac{\log R}{\log(R + s(R)}.$$

So

$$\begin{aligned} \overline{\tau}(X) = \overline{\lim} \frac{\log \gamma(\overline{B}(x_0, R))}{\log(R} &\geq \overline{\lim} \frac{\log \gamma(\overline{B}(x_0, R + s(R)))}{\log(R + s(R)} \\ &\geq \overline{\lim} \frac{\log s(R)}{\log R} \frac{\log R}{\log(R + s(R)} = \sigma. \end{aligned}$$

□

The following modification of Example 3.1 shows that a sparse space's tryposity order at infinity may exceed its sparsity order.

Example 3.7. Let $X = X_3 \cup \{n^3 + n : n = 1, 2, 3, \dots\}$. Then $\tau(X) = \tau(X_3) = 2/3$ but $\sigma(X) = 1/3$.

In fact, a space may not even be quasi-sparse but still have tryposity order at infinity 1, as this modification of Example 3.2 shows:

Example 3.8. Let $X = \{0\} \cup \bigcup_{k=1}^{\infty} [2^{2k}, 2^{2k+1}]$. Then $\tau(X) = 1$ as in Example 3.2, but X is not quasi-sparse.

4. Sparse ends

We want to define analogues of topological ends in metric spaces that are not connected or even coarsely connected, like sparse spaces.

Let (X, d) be a metric space with a base point x_0. For $\alpha \in [0, 1]$ and $K > 0$ we define the relation of being (α, K)-*close* (relative to the base point x_0). Namely, $x, y \in X$ are in this relation if

$$d(x, y) \leq K[\min\{d(x, x_0), d(y, x_0)\}]^\alpha + K.$$

Let S be a subset of X. We say x, y are (α, K)-*chained* (in S) if there are points $x_1, x_2, \ldots, x_n$ (for some $n > 1$) in S such that $x_1 = x$, $x_n = y$, and x_i, x_{i+1} are (α, K)-close for $i = 1, 2, \ldots, n-1$. This is of course an equivalence relation.

We call the equivalence classes the (α, K)-*components* of S. If S has just one (α, K)-component, we call it (α, K)-*connected*. If S is (α, K)-connected for some $K > 0$, we call it α-*connected*.

Remark 4.1. (i) The (α, K)-components of S are the maximal (α, K)-connected subsets of S.
(ii) If S is (α, K)-connected, then it is (α, K')-connected for all $K' \geq K$.
(iii) If $S = A \cup B$ is (α, K)-connected, with $A, B \neq \emptyset$, then there are (α, K)-close $x \in A, y \in B$.
(iv) If A and B are (α, K)-connected subsets of S and some $x \in A$ and $y \in B$ are (α, K)-close, then $A \cup B$ is (α, K)-connected.

Remark 4.2. Let $X = \{x_0, x_1, x_2, \ldots\}$ be the set of terms of an increasing sequence in the halfline $[0, \infty)$ going to infinity. Then X is (α, K)-connected if and only if all pairs of consecutive elements $x_i, x_{i+1} \in X$ are (α, K)-close.

If $A \subset X$, we write $d(A, x_0)$ for $\inf\{d(x, x_0) | x \in A\}$.

Lemma 4.3. *Let $R > 0$. A subset S of a pointed metric space (X, x_0, d) is α-connected if and only if $S \setminus B(x_0, R)$ is α-connected.*

More specifically: if S is (α, K)-connected, then $S \setminus B(x_0, R)$ is (α, K')-connected, where $K' = 2K(R^\alpha + 1) + 2R$, and on the other hand if $S \setminus B(x_0, R)$ is (α, K)-connected, then S is (α, K')-connected, where $K' = \max\{K, 2d(S \setminus B(x_0, R), x_0) + 1\}$.

Proof. Assume first that S is (α, K)-connected. We show that every $x, y \in S \setminus B(x_0, R)$ are (α, K')-chained in $S \setminus B(x_0, R)$, where $K' = 2K(R^\alpha + 1) + 2R$.

Note that it suffices to show this when each z other than x and y in the (α, K)-chain connecting them is in $B(x_0, R)$.

Let this (α, K)-chain be $x, x', \ldots, y', y$. Then

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &\leq K d(x', x_0)^\alpha + K + 2R + K d(y', x_0)^\alpha + K \\ &\leq 2KR^\alpha + 2K + 2R \\ &\leq K' \end{aligned}$$

so that x and y are (α, K')-close.

Now instead assume that $S \setminus B(x_0, R)$ is (α, K)-connected.

Let $M = d(S \setminus B(x_0, R), x_0)$, so that $M \geq R$, and pick $y_0 \in S \setminus B(x_0, R)$ with $d(x_0, y_0) \leq M + 1$. Set $K' = \max\{K, 2M + 1\}$. Consider $x, y \in S$.

If $x, y \in S \setminus B(x_0, R)$, then they are (α, K)-chained (in $S \setminus B(x_0, R)$) and thus (α, K')-chained.

If $x, y \in S \cap B(x_0, R)$, then they are $(\alpha, 2R)$-close and thus (α, K')-close.

If $x \in S \cap B(x_0, R)$ and $y \in S \setminus B(x_0, R)$, then they are (α, K')-chained (in S) since $d(x, y_0) \le 2M + 1$ so that x is (α, K')-close to y_0. □

Note that the first half of the proof shows that if every $x, y \in S \setminus B(x_0, R)$ are (α, K)-chained in S, then they are (α, K')-chained in $S \setminus B(x_0, R)$ (so that $S \setminus B(x_0, R)$ is (α, K')-connected).

Proposition 4.4. *Quasi-isometries preserve α-connectedness. That is, if S is an α-connected subset of a pointed metric space (X, x_0, d) and $\varphi : (X, x_0, d) \to (X', x_0', d')$ is a quasi-isometry to another pointed metric space, then $S' = \varphi(S)$ is α-connected. More specifically, if S is (α, K)-connected, then S' is (α, K')-connected, where $K' = \max\{C(K+1), 2^\alpha C^{1+\alpha} K\}$ for a constant C associated with the quasi-isometry.*

Proof. Since φ is a quasi-isometry, there is $C > 1$ such that for all $x, y \in S$, we have

$$d'(\varphi(x), \varphi(y)) \le C d(x, y) + C$$

and

$$d(x, y) \le C d'(\varphi(x), \varphi(y)) + C.$$

By the Lemma, it suffices to show that $S' \setminus B(x_0', 2C^2 + C)$ is α-connected. Again by the Lemma, there is some $K > 0$ so that $S \setminus B(x_0, 2C)$ is (α, K)-connected.

Let $x', y' \in S' \setminus B(x_0', 2C^2 + C)$ and let $K' = \max\{C(K+1), 2^\alpha C^{1+\alpha} K\}$. By the remark after the Lemma, it suffices to show that x', y' are (α, K')-chained in S' (since then they will also be (α, K')-chained in $S' \setminus B(x_0', 2C^2 + C)$).

Note first that $\varphi(B(x_0, 2C)) \subset B(x_0', 2C^2 + C)$. So there are $x, y \in S \setminus B(x_0, 2C)$ with $\varphi(x) = x', \varphi(y) = y'$ and so also an (α, K)-chain $x = x_1, \dots, x_n = y$ in $S \setminus B(x_0, 2C)$. Let $x_i' = \varphi(x_i)$ for each i. So for each i, $x_i' \notin B(x_0', 1)$ since $x_i \notin B(x_0, 2C)$. Then for each $i = 1, \dots, n-1$ we have

$$\begin{aligned}
d'(x_i', x_{i+1}') &\le C d(x_i, x_{i+1}) + C \\
&\le C(K[\min\{d(x_i, x_0), d(x_{i+1}, x_0)\}]^\alpha + K) + C \\
&= CK[\min\{d(x_i, x_0), d(x_{i+1}, x_0)\}]^\alpha + C(K+1) \\
&\le CK[C \min\{d'(x_i', x_0'), d'(x_{i+1}', x_0')\} + C]^\alpha + C(K+1) \\
&= C^{1+\alpha} K[\min\{d'(x_i', x_0'), d'(x_{i+1}', x_0')\} + 1]^\alpha + C(K+1) \\
&\le C^{1+\alpha} K[2 \min\{d'(x_i', x_0'), d'(x_{i+1}', x_0')\}]^\alpha + C(K+1) \\
&= 2^\alpha C^{1+\alpha} K[\min\{d'(x_i', x_0'), d'(x_{i+1}', x_0')\}]^\alpha + C(K+1) \\
&\le K'[\min\{d'(x_i', x_0'), d'(x_{i+1}', x_0')\}]^\alpha + K'
\end{aligned}$$

So x' and y' are (α, K')-chained in S' as we wanted. □

Example 4.5. Although by Proposition 4.4 a bi-Lipschitz image of an α-connected set is α-connected, this example shows that if X is α-connected and $f : X \to Y$ is Lipschitz (even with Lipschitz constant 1), then it may happen that $f(X)$ is not α-connected (with the same α).

Set $X = \{i^2 : i = 0, 1, 2, 3, \dots\}$ and $\alpha = 1/2$. Let us construct f and $Y = f(X)$ by induction. Our set Y will be of the form $\{y_n : n = 0, 1, 2, 3, \dots\}$ with $y_0 \le y_1 \le y_2 \le y_3 \dots$. We start with $y_0 = 0$. Suppose that y_i is defined and $f(i^2) = y_i$ for $i = 0, 1, 2, \dots, n$. If k is sufficiently large, we have

$$2k - 1 = d\big(k^2, (k-1)^2\big) > n\big(d(y_0, y_n)\big)^\alpha + n = n\sqrt{y_n} + n.$$

We fix such $k > n+1$ and set $y_i = f(i^2) = y_n$ for $i = n+1, n+2, n+3, \ldots, k-1$ and $y_k = f(k^2) = y_n + k^2 - (k-1)^2$. Then y_{k-1} and y_k are not (α, n)-close. By Remark 4.2, $f(X)$ is not (α, n)-connected. Since this happens for infinitely many n, $f(X)$ is not α-connected. $\Diamond$

Corollary 4.6. *Let S be a subset of a metric space (X, d) and $x_0, y_0 \in X$. Then if S is α-connected relative to x_0 then it is α-connected relative to y_0.*

More specifically, if S is (α, K)-connected relative to x_0, then it is (α, K')-connected relative to y_0, where $K' = \max\{C(K+1), 2^\alpha C^{1+\alpha} K\}$ and $C = d(x_0, y_0)$.

Proof. Let $\varphi : (X, x_0) \to (X, y_0)$ be the quasi-isometry taking x_0 to y_0 and leaving every other point fixed. $\square$

Given an unbounded subset S of a pointed metric space (X, x_0, d), let us look at the set of unbounded (α, K)-components of $S \setminus B(x_0, R)$, provided with the discrete topology. For $R < R'$, each unbounded (α, K)-component for R' is contained in some unbounded (α, K)-component for R. However, "splitting" can occur, so the number of unbounded (α, K)-components for R' can be larger than for R, but it cannot be smaller.

Let the *real* (α, K)*-ray* be the (barely) (α, K)-connected sequence of reals $\mathbb{R}_{(\alpha,K)} = \{(a_n)_{n=0}^\infty | a_0 = 0, a_n = a_{n-1} + K a_n^\alpha + K, n > 0\} \subset \mathbb{R}_{\geq 0}$.

The image in $S \subset X$ of $\mathbb{R}_{(\alpha,K)}$ under a quasi-isometry is (α, K')-connected for K' as in the proposition. We define an (α, K)*-quasi-ray* in $S \subset X$ as an (α, K)-connected sequence in S that is the image of some real (α, K')-ray $\mathbb{R}_{(\alpha,K')}$ under a quasi-isometry, and we say that the (α, K)-quasi-ray is *based* at the image of 0.

Note that if (x_n) is an (α, K)-quasi-ray, then for any $R > 0$, $(x_n) \setminus B(x_0, R)$ has exactly one unbounded (α, K)-component.

If (x_n) and (y_n) are two (α, K)-quasi-rays in S, set $(x_n) \sim (y_n)$ if for all $R > 0$, the unbounded (α, K)-component of $(x_n) \setminus B(x_0, R)$ and of $(y_n) \setminus B(x_0, R)$ lie in the same (unbounded) (α, K)-component of $S \setminus B(x_0, R)$.

We define an (α, K)*-end* A as an equivalence class of (α, K)-quasi-rays.

A basic open set of (α, K)-ends is a set of equivalence classes of representative sequences (x_n) such that for some $R > 0$, the unbounded (α, K)-component of each $(x_n) \setminus B(x_0, R)$ lies in the same (unbounded) (α, K)-component of $S \setminus B(x_0, R)$.

For example, if X is the binary tree with root x_0 viewed as a metric space, S is the vertex set of X, and we take $\alpha = 0$, the set of (α, K)-ends of S is empty for $K < 1$ while for $K \geq 1$ it corresponds to the set of infinite paths in X topologized as a Cantor set, i.e. the usual space of topological ends of X.

Clearly (α, K)-ends aren't invariant under quasi-isometries, as (α, K)-connectedness isn't. But we use them to define α-ends, which will be invariant.

Now we keep α constant, but vary K. Here we do not have monotonicity. As K grows, two (α, K)-ends can be glued together, but also new (α, K)-ends can be created. Thus, we will take a direct limit over K to define the space of α-ends.

First note that if $K < K'$, then every (α, K)-end is contained in some (α, K')-end. Thus there is a well defined map from (α, K)-ends to (α, K')-ends.

Fix α. For each $K > 0$, let $\mathcal{E}_K^\alpha$ be the set of all (α, K)-ends. We define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{K>0} \mathcal{E}_K^\alpha$: if $E \in \mathcal{E}_K^\alpha$ and $E' \in {\mathcal{E}^\alpha}_{K'}$ then $E \sim E'$ if there is some $L \geq K, K'$ and some $F \in \mathcal{E}_L$ such that E and E' are contained in F.

An equivalence class of this relation will be called an α-*end*. That is, the set of α-ends of the subset S of the pointed metric space (X, x_0, d) is

$$\mathcal{E}^\alpha_\infty = \mathcal{E}^\alpha_\infty(S) = \bigsqcup_{K>0} \mathcal{E}^\alpha_K / \sim .$$

So every $e \in \mathcal{E}^\alpha_\infty$ is an equivalence class $e = [E]$, where $E \in \mathcal{E}^\alpha_K$ for some K.

We endow the set of α-ends with the direct limit topology, so that a set of α-ends is open when for each K the corresponding set of (α, K) representatives is.

Proposition 4.7. *A quasi-isometry induces a homeomorphism of the spaces of α-ends.*

That is, if $\alpha \geq 0$ and S is a subset of a pointed metric space (X, x_0, d) with α-ends $\mathcal{E}^\alpha_\infty(S)$ and $\varphi : (X, x_0, d) \to (X', x'_0, d')$ is a quasi-isometry to another pointed metric space, then $\mathcal{E}^\alpha_\infty(S)$ is homeomorphic to $\mathcal{E}^\alpha_\infty(S')$, where $S' = \varphi(S)$.

Proof. We first show that φ induces a bijection of the set of α-ends. By symmetry, it suffices to show that φ induces a map from α-ends of S to α-ends of S' and then check that this map is one to one.

If e is an α-end of S, then $e = [E]$, where $E \in \mathcal{E}^\alpha_K(S)$ for some K. Then it follows from Proposition 4.4 that φ takes E to some $E' \in \mathcal{E}^\alpha{}_{K'}(S')$, where K' is as given in the proposition. So the induced map φ_* on α-ends takes $e \in \mathcal{E}^\alpha_\infty(S)$ to $e' \in \mathcal{E}^\alpha_\infty(S')$, where $e' = [E']$.

Now suppose that e_1 and e_2 are α-ends of S and $\varphi_*(e_1) = \varphi_*(e_2) = e'$. Let $\psi : (X', x'_0, d') \to (X, x_0, d)$ be an inverse quasi-isometry of φ with $\psi(S') \subset S$ and $\psi(S')$ K_0-dense in S, and let $e = \psi_*(e')$. If $e_1 = [E_{K_1}]$ and $e_2 = [E_{K_2}]$, and we take E_K with $e = [E_K]$ and $K \geq \max(K_1, K_2)$, then E_{K_1} and E_{K_2} are contained in E_K so we have that $e_1 = e_2$. So φ_* is a bijection on the set of α-ends.

It remains to prove that φ_* is a homeomorphism. By symmetry, it suffices to show that φ_* is continuous. We show that φ_* is continuous at every α-end e of S.

Let $e = [E]$, where $E \in \mathcal{E}^\alpha_K(S)$ for some K. Let $e' = \varphi_*(e) = [E']$, where $E' \in \mathcal{E}^\alpha{}_{K'}(S')$ for $K' \geq K'_0$, where K'_0 is the constant given in Proposition 4.4.

A basic neighborhood V of the α-end e' corresponds to the unbounded (α, K')-connected component of $S' \setminus B(x'_0, R')$ containing e' for some R'. We find a basic neighborhood U of e mapped by φ into V. Let $R = CR' + C$, where C is the constant in Proposition 4.4. Then $\varphi(X \setminus B(x_0, R)) \subset X' \setminus B(x'_0, R')$ since $\varphi(B(x_0, R)) \supset B(x'_0, R')$. So $\varphi(S \setminus B(x_0, R)) \subset S' \setminus B(x'_0, R')$. Let U be the basic neighborhood of e corresponding to the unbounded (α, K)-connected component of $S \setminus B(x_0, R)$ containing e. Then $\varphi(U)$ is an (α, K')-connected subset of $S' \setminus B(x'_0, R')$, and it contains e', so it is contained in V. $\square$

Corollary 4.8. *The space of α-ends is invariant under change of basepoint.*

Remark 4.9. In light of Proposition 4.7, for $\alpha \geq 0$ we can define the space of α-ends of a quasi-sparse space as the space of α-ends of any of its sparse skeletons.

Remark 4.10. Our 0-connectedness is just coarse connectedness (e.g. [CH]), i.e. Gromov's [G] long-range connectedness. Our 0-ends coincide with the coarse ends of Alvarez Lopez and Candel [AC].

5. Examples

Let us provide some examples.

Example 5.1. Let X be a subset of $[0,\infty)$, consisting of points n^q, where $q \geq 1$ is a constant, and $n = 0, 1, 2, \dots$. Set $x_0 = 0$ as the base point. Fix $\alpha \in [0,1]$ and $K > 0$. If $n > 0$, then n^q and $(n+1)^q$ are (α, K)-close if and only if

$$(n+1)^q - n^q \leq K n^{q\alpha} + K. \tag{5.1}$$

We have

$$\begin{aligned}\lim_{x\to\infty} \frac{(x+1)^q - x^q}{x^{q-1}} &= \lim_{x\to\infty} \frac{(1+\frac{1}{x})^q - 1}{\frac{1}{x}} = \lim_{y\to 0} \frac{(1+y)^q - 1}{y} \\ &= \frac{d}{dy}\big((1+y)^q\big)\Big|_{y=0} = q(1+y)^{q-1}\Big|_{y=0} = q.\end{aligned}$$

Therefore, (5.1) holds for large K and n if and only if $q - 1 \leq q\alpha$, that is, if and only if $\alpha \geq (q-1)/q$. We see easily that X has no α-ends if $\alpha < (q-1)/q$ and has one α-end if $\alpha \geq (q-1)/q$.

In particular, if we take $q = 2$ we see that the set of squares $X = \{n^2 : n = 0, 1, 2, 3, \dots\}$ has one α-end if $\alpha \geq 1/2$ and no α-ends if $\alpha < 1/2$. ♢

Remark 5.2. White and Willett ([WW], Example 3.3) describe a Cartan subalgebra of the uniform Roe algebra of the set of squares (or any sparse set of nonnegative integers).

Example 5.3. Let us modify the preceding example. Namely, instead of the half-line (which now can be viewed as one ray starting at the origin in $\mathbb{R}^2$) we take M rays starting at the origin. We set the origin as the base point x_0. On the mth ray we take as elements of X points whose distance from x_0 is n^{q_m}, where $n = 0, 1, 2, \dots$. Here $q_1, \dots, q_M$ are constants larger than or equal to 1.

If K and α are fixed and R is large enough, no points from different rays outside $B(x_0, R)$ are (α, K)-close. Therefore, by the same arguments as in Example 5.1, we see that the number of α-ends of X is equal to the number of those m for which $(q_m - 1)/q_m \leq \alpha$. ♢

Example 5.4. We can modify our example even further. Namely, instead of finitely many rays, we can take infinitely many, such that in the intersection of their union with the unit circle every point is isolated. Then clearly the number of α-ends of X is at least the number of those m for which $(q_m - 1)/q_m \leq \alpha$. ♢

Example 5.5. Take points $z_m = (\cos(1/m), \sin(1/m))$ on the unit circle in $\mathbb{R}^2$, where $m = 1, 2, 3, \dots$. Define $X = \{n^{q_m} z_m : m, n = 1, 2, 3, \dots\} \cup \{(0,0)\}$ for some sequence $(q_m), q_m > 1$. This is the situation from the preceding example, and we see X has at least zero 0-ends, which gives us no information. When we look along the rays, we cannot get a better estimate. Yet we can define the sequence (q_m) in such a way that X will have at least one 0-end.

We do it in the following way. We partition the sequence $(1/m)$ into infinitely many subsequences. Then we take a bijection from the set of subsequences onto the set of natural numbers larger than 1. For each subsequence (y_k) we take numbers a_k such that 2^{a_k} converges to the natural number corresponding to the subsequence (y_k). In such a way the set $\{(n,0) : n = 2,3,4,\dots\}$ belongs to the closure of X. Therefore, X has at least one 0-end, which we see when going along this set to infinity. This is a subset of the horizontal ray, but no points of this ray belong to X, so we cannot use Example 5.4 to see this end. $\Diamond$

Example 5.6. Denote the space from Example 5.1 by X_q. Now take the set $X = X_{q_1} \times X_{q_2}$ for some $q_1, q_2 \geq 1$. If $\alpha \geq (q_i - 1)/q_i$ for $i = 1, 2$, then X is α-connected, so it has 1 α-end. If $\alpha \geq (q_1 - 1)/q_1$, but $\alpha < (q_2 - 1)/q_2$ (or vice versa), then we can view X as the disjoint union of spaces isometric to X_{q_1} (or X_{q_2}) with no connection between them. This means that X has infinitely (countably) many α-ends. Finally, if $\alpha < (q_i - 1)/q_i$ for $i = 1, 2$, then X has no α-ends. $\Diamond$

Example 5.7. As in Example 5.5, we take points $z_m = (\cos(1/m), \sin(1/m))$ on the unit circle in $\mathbb{R}^2$, where $m = 1, 2, 3, \dots$, but now also take $z_\infty = (1, 0)$. Now we define $X = \{nz_m | m \leq \infty, n = 0, 1, 2, 3, \dots\}$. Then X is coarsely connected (in fact (0,1)-connected), and its space of 0-ends is its space of (0,1)-ends, which is homeomorphic to the subspace $Z = \{z_m | m \leq \infty\}$ of the circle. $\Diamond$

Example 5.8. As in Example 5.7, we take $Z = \{z_m | m \leq \infty\}$ where $z_m = (\cos(1/m), \sin(1/m))$ on the unit circle in $\mathbb{R}^2$ for $m = 1, 2, 3, \dots$ and $z_\infty = (1, 0)$. Let $X = \{nmz_m | m = 1, 2, 3, \dots, n = 0, 1, 2, \dots\} \cup \{nz_\infty | n = 0, 1, 2, \dots\}$. Then X is coarsely connected (though not (0,K)-connected for any K), and, in contrast to the previous example, its space of 0-ends is a countable discrete space, which can be thought of as Z but with the discrete topology. $\Diamond$

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